# LATTICE-ORDERED CONDITIONAL INDEPENDENCE MODELS FOR MISSING DATA

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$$X = (X_1, \dots, X_p) \sim N_p(\underbrace{\mu, \Sigma}_{\text{UNKNOWN}})$$

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$X^{(1)}, \dots, X^{(n)}$ RANDOM SAMPLE FROM MULTIVARIATE NORMAL POPULATION.
$(\mu, \Sigma)$ unknown, to be estimated

① COMPLETE SAMPLE:

$X^{(1)}$: 1 2 3
$X^{(2)}$: 1 2 3
1 2 3
⋮ 1 2 3
$X^{(n)}$: 1 2 3

DATA PATTERN:

$\mathcal{S} = \{123\}$

COMPLETE

② INCOMPLETE SAMPLE:

$X^{(1)}$: 1 2 3
$X^{(2)}$: 1
1 2
⋮ 1 2 3
1 2
1
$X^{(n)}$: 1 2 3

(MISSING OBS'S)

DATA PATTERN:

$\mathcal{S} = \{1, 12, 123\}$

INCOMPLETE, MONOTONE

③

1 2 3
1 2
1 2
1 3
1 2 3
1 3

$\mathcal{S} = \{12, 13, 123\}$

INCOMPLETE, NON-MONOTONE

$I = \{1, 2, 3, \cdots, p\}$ (= index set)

A DATA PATTERN $\mathcal{S}$ is ANY collection of NON-EMPTY SUBSETS of I. $[\mathcal{S} \subseteq 2^{I \setminus \emptyset}]$

→ $\mathcal{S}$ is a partially ordered SET UNDER INCLUSION $\subseteq$

→ $2^I$ is a FINITE DISTRIBUTIVE LATTICE: CLOSED UNDER $\cap$ + $\cup$.

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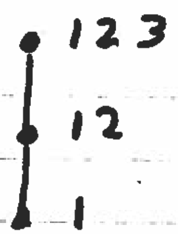
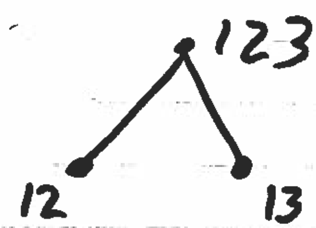
MONOTONE $\mathcal{S}$:




(TOTALLY ORDERED
CHAIN
NESTED)

(INCLUDES COMPLETE CASE)

NON-MONOTONE $\mathcal{S}$:




(NOT TOTALLY ORDERED)

GOAL #1: $\boxed{\text{FIND MLE's } \hat{\mu}, \hat{\Sigma}}$

IN MONOTONE CASE:

→ FACTORIZATION OF LIKELIHOOD F'n
FACTORIZATION OF PARAMETER SPACE

⟹ MODEL IS THE PRODUCT OF ORDINARY (MULTIVARIATE NORMAL) LINEAR REGRESSION MODELS

⟹ EXPLICIT SOLUTIONS FOR MLE's OF THE REGRESSION PARAMETERS

⟹ EXPLICIT SOLUTIONS FOR $\hat{\mu}, \hat{\Sigma}$.

(TWA (1957 JASA), RAO, BARTLETT?, WILKS?...)

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NO SUCH FACTORIZATION IN NON-MONOTONE CASE,
(except in special cases)

# EXAMPLE: MONOTONE $\mathcal{S}$

$$1\,2\,3 \longrightarrow f(123) = f(1)\,f(2|1)\,f(3|12)$$
$$1\,2 \longrightarrow f(12) = f(1)\,f(2|1)$$
$$1 \longrightarrow f(1) = f(1)$$

$$\therefore f(\text{DATA}) = \underbrace{f(1)\,f(1)\,f(1)}\;\underbrace{f(2|1)\,f(2|1)}\;\underbrace{f(3|12)}$$
$$= f(1) \times f(2|1) \times f(3|12)$$

$$\begin{cases} f(1) \text{ depends on } \mu_1,\ \Sigma_{11} \\ f(2|1) \quad \text{"} \quad \text{"} \quad \mu_{2\cdot1},\ \beta_{2\cdot1},\ \Sigma_{22\cdot1} \\ \qquad\qquad [\mu_2 - \Sigma_{21}\Sigma_{11}^{-1}\mu_1]\ [\Sigma_{21}\Sigma_{11}^{-1}]\ [\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}] \\ f(3|12) \quad \text{"} \quad \text{"} \quad \mu_{3\cdot12},\ \beta_{3\cdot12},\ \Sigma_{33\cdot12} \\ \qquad\qquad [\cdots]\ [\cdots]\ [\cdots] \end{cases}$$

$\{f(2|1), f(3|12)$ are LF's OF ORD.$^y$ (MULT.) LINEAR REGRES. MODELS

AND: THE PARAMETER SPACE FACTORS:

$$(\mu, \Sigma) \in \underline{\mathbb{R}^p \times \mathcal{P}_p^+} \quad (= \text{all } p \times p \text{ pos. def. mxs})$$

$$\Longleftrightarrow \Big( (\mu_1, \Sigma_1), (\mu_{2\cdot 1}, \beta_{2\cdot 1}, \Sigma_{22\cdot 1}), (\mu_{3\cdot 12}, \beta_{3\cdot 12}, \Sigma_{33\cdot 12}) \Big)$$

$$\in \underline{\left(\mathbb{R}^{p_1} \times \mathcal{P}_{p_1}^+\right) \times \left(\mathbb{R}^{p_2} \times M(p_2 \times p_1) \times \mathcal{P}_{p_2}^+\right)}$$
$$\underline{\times \left(\mathbb{R}^{p_3} \times M(p_3 \times (p_1 + p_2)) \times \mathcal{P}_{p_3}^+\right)}$$

$\therefore$ CAN SOLVE THE INDIVIDUAL (CONDITIONAL) REGRESSION PROBLEMS <u>SEPARATELY</u>, THEN COMBINE THE REGRESSION ESTIMATES TO RECONSTRUCT $(\hat{\mu}, \hat{\Sigma})$. - <u>explicit</u>

(non-iterative,

NON-MONOTONE PATTERNS? (TWA 1957, D. RUBIN 1987)

$$S = \left\{ \begin{matrix} 1 & 2 & 3 \\ 1 & 2 & \\ 1 & & 3 \end{matrix} \right\} \begin{matrix} \rightsquigarrow & f(1)\, f(2|1)\, f(3|12) \\ \rightsquigarrow & f(1)\, f(2|1) \quad\quad\;\; \updownarrow \\ \rightsquigarrow & f(1) \quad\quad\quad\quad f(3|1) \end{matrix}$$

DIFFERENT! CANNOT COMBINE!

LF does NOT FACTOR

MLE VIA EM ALGORITHM (ITERATIVE - MAY NOT CONVERGE!)

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RUBIN NOTED: IF we ASSUME

$$X_2 \perp\!\!\!\perp X_3 \mid X_1 \quad \leftrightarrow \quad 2 \perp\!\!\!\perp 3 \mid 1$$

then $\boxed{f(3|12) = f(3|1)}$, LF does FACTOR, can obtain MLE explicitly (non-iterative)

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∴ Restrict $\Sigma$ via a C.I. assumption

- C.I. Assumption can be tested
- At least, gives reasonable starting value for EM.

[NO PAGES 8, 9]

## ANOTHER NON-MONOTONE PATTERN:

$$
\mathcal{S}=\left\{\begin{array}{ccccc} 1&2&3&4&5\\ 1&&3&&\\ 1&2&&4&\\ 1&2&&&5 \end{array}\right\}
\begin{array}{l}
\rightsquigarrow f(1)\,f(2|1)\,f(3|12)\,f(4|123)\,f(5|1234)\\
\rightsquigarrow f(1)\qquad\quad\;\; f(3|1)\\
\rightsquigarrow f(1)\,f(2|1)\qquad\qquad\quad f(4|12)\\
\rightsquigarrow f(1)\,f(2|1)\qquad\qquad\qquad\qquad\quad f(5|12)
\end{array}
$$

$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{\text{DIFFERENT - CANNOT COMBINE}}$

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HOW DETERMINE THE <u>MINIMAL</u> C.I. ASSUMPTION s.t. LF FACTORS?

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CAN ASSUME COMPLETE INDEP:

$$1 \perp\!\!\!\perp 2 \perp\!\!\!\perp 3 \perp\!\!\!\perp 4 \perp\!\!\!\perp 5$$

$$f(12345) = f(1)\,f(2)\,f(3)\,f(4)\,f(5)$$

BUT THIS IS MUCH TOO STRONG.

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$\exists$ a <u>MORE</u> <u>PARSIMONIOUS</u> C.I. ASSUMPTION S.T. LF FACTORS!

$$3 \perp\!\!\!\perp 4 \perp\!\!\!\perp 5 \,|\, 12 \quad \text{and} \quad 2 \perp\!\!\!\perp 3 \,|\, 1$$

$$f(12345) = f(1)\,f(2|1)\,f(3|1)\,f(4|12)\,f(5|12)$$

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<u>IN GENERAL:</u> $\mathcal{S} \rightsquigarrow \mathcal{K}(\mathcal{S}) \rightsquigarrow$ "LCI" model $\mathcal{N}(\mathcal{K})$.

# STUDY:
# STRUCTURE OF FINITE DISTRIB. LATTICES

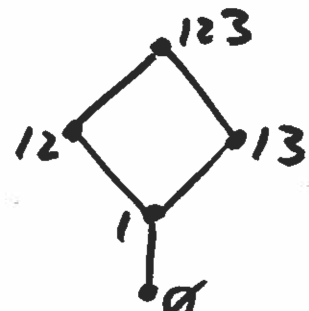
$\mathcal{S} \subseteq 2^I$ (ANOVA TABLE)

$\mathcal{K} = \mathcal{K}(\mathcal{S}) \subseteq 2^I$

= LATTICE (RING) GENERATED BY $\mathcal{S}$

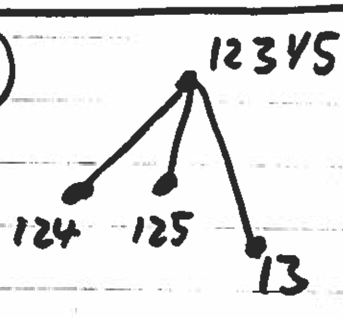
$\mathcal{K} = \{$all $\cap$'s & $\cup$'s of elts of $\mathcal{S}\} \cup \{\emptyset\}$

I.

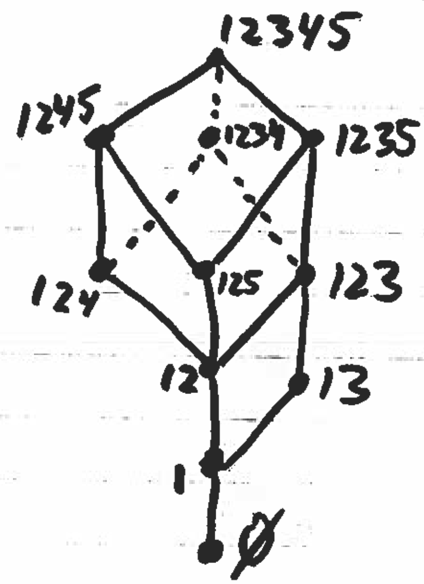




$\mathcal{S} \longrightarrow \mathcal{K}(\mathcal{S})$

II.




$\longrightarrow$



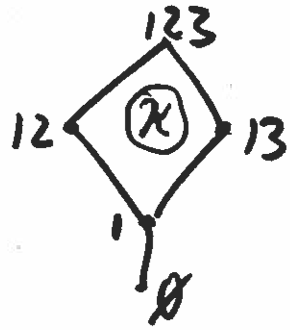

$\mathcal{S} \longrightarrow \mathcal{K}(\mathcal{S})$

$\mathcal{K}$ CLOSED UNDER $\cap$ & $\cup$ (& NOT)

→ $\mathcal{K}$ determines A SET OF C.I. conditions:

REQUIRE: (*) $\boxed{X_K \perp\!\!\!\perp X_L \mid X_{K\cap L} \quad \forall K,L\in\mathcal{K} \ (K,L\subseteq I)}$

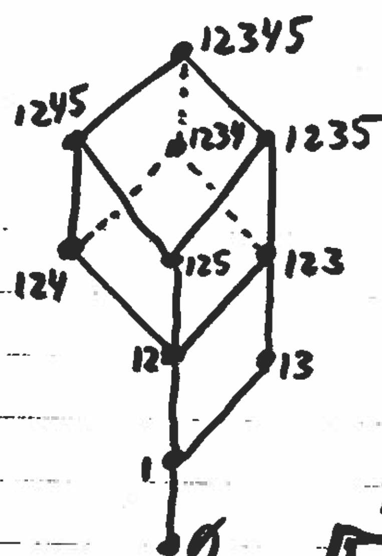
(I.)




$X_{12}\equiv(X_1,X_2) \perp\!\!\!\perp X_{13}\equiv(X_1,X_3) \mid X_1$

EQUIVALENTLY: $\boxed{X_2 \perp\!\!\!\perp X_3 \mid X_1}$

(II)




| | | |
|---|---|---|
| $3 \perp\!\!\!\perp 4 \mid 125$ | $3 \perp\!\!\!\perp 4 \mid 12$ | $2 \perp\!\!\!\perp 3 \mid 1$ |
| $3 \perp\!\!\!\perp 5 \mid 124$ | $3 \perp\!\!\!\perp 5 \mid 12$ | |
| $4 \perp\!\!\!\perp 5 \mid 123$ | $4 \perp\!\!\!\perp 5 \mid 12$ | |

EQUIVALENTLY:

$\boxed{3 \perp\!\!\!\perp 4 \perp\!\!\!\perp 5 \mid 12, \qquad 2 \perp\!\!\!\perp 3 \mid 1}$

[(*) TRIVIAL WHEN $K\subseteq L$, SO DON'T GET SO MANY RESTRICTIONS]

LET $\mathcal{P}^{+}_{\mathcal{K}} = \left\{ \Sigma \text{ pos. det.} \;\middle|\; \begin{array}{l}\Sigma \text{ SATISFIES C.I.} \\ \text{CONDITIONS FROM } \mathcal{K}\end{array} \right\}$
$P(\mathcal{K})$

<u>NOTE</u>: $\mathcal{S}$ MONOTONE CHAIN $\Leftrightarrow \mathcal{K}(\mathcal{S}) = \mathcal{S}$

$\Longleftrightarrow \mathcal{P}^{+}_{k} = \{$ <u>All</u> $\Sigma$, <u>no</u> RESTRICTIONS $\}$

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If $\mathcal{S}$ is NON-MONOTONE, generate $\mathcal{K}(\mathcal{S})$, <u>reduce</u> <u>PARAMETER</u> <u>SPACE</u> TO $\mathbb{R}^{p} \times \mathcal{P}^{+}_{\mathcal{K}}$.
$(\mu, \Sigma)$

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<u>KEY</u> <u>THEOREM 1</u>: THE LF + (REDUCED) PARAMETER SPACE FACTOR. THE MODEL BECOMES A <u>PRODUCT</u> OF ORDINARY (MULT. NORMAL) LINEAR REGRESSION MODELS.

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WHAT ARE THESE FACTORIZATIONS?

[NO PAGES <u>14</u>, <u>15</u>]

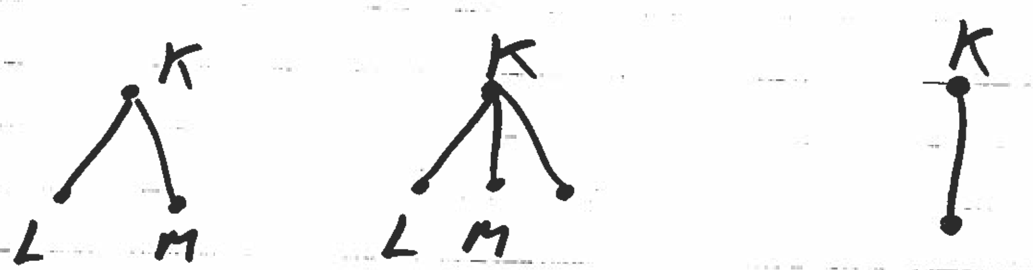
$$\mathcal{S} \xrightarrow{\text{generate}} \mathcal{K}(\mathcal{S}).$$

(NOW FORGET ABOUT ORIGINAL $\mathcal{S}$.)

LET $\mathcal{J}(\mathcal{K}) = \{$all non-empty join-irreducible elt's of $\mathcal{K}\}$:

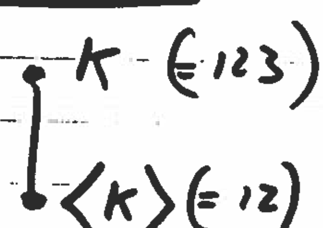
$K$ is join-IRREDUCIBLE $\iff$

$$\boxed{K = L \cup M \Rightarrow K = L \text{ or } K = M}$$




NO NO YES!

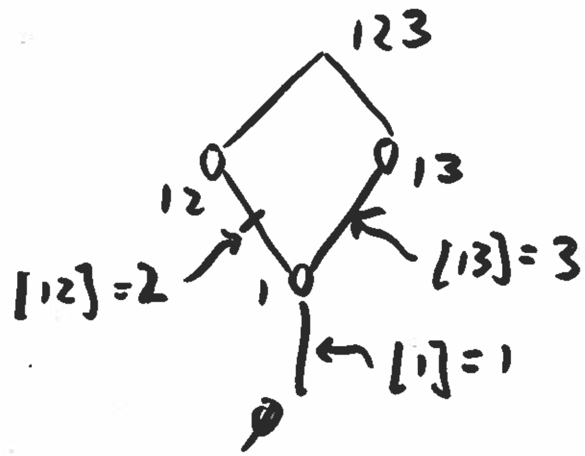
$K$ JOIN-IRREDUCIBLE $\Rightarrow \exists$ UNIQUE element $\langle K \rangle \in \mathcal{K}$ COVERED BY $K$:




DEFINE $[K] = K \setminus \langle K \rangle \; (=3)$

NOTE: $[K] \notin \mathcal{K}$

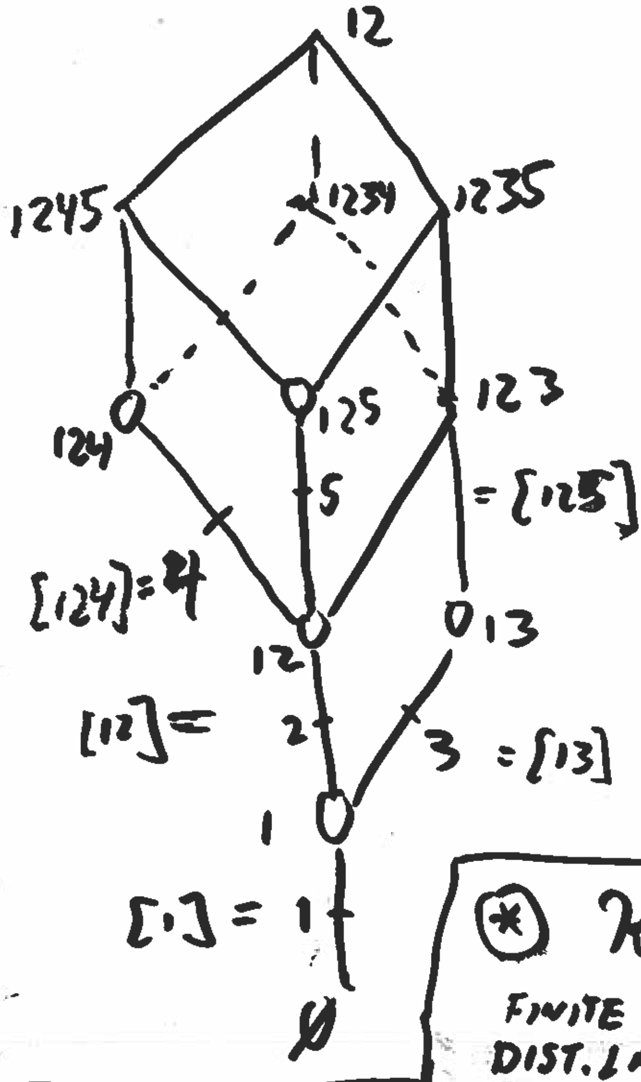
BASIC DECOMPOSITION: $I = \dot{\bigcup}\{[K] \mid K \in \mathcal{J}(\mathcal{K})\}$




$J(\mathcal{K}) = \{12,\ 13,\ 1\}$

$I = [12] \dot{\cup} [13] \cdot [1]$

123   2   3   1




$J(\mathcal{K}) = \{124, 125, 12, 13, 1\}$

| | | |
|---|---|---|
| $I =$ | $[124]$ | 4 |
| 12345 | $\dot{\cup}[125]$ | 5 |
| | $\dot{\cup}[12]$ | 2 |
| | $\dot{\cup}[13]$ | 3 |
| | $\dot{\cup}[1]$ | 1 |

$(*)$ $\mathcal{K} \Rightarrow J(\mathcal{K})$ (MISSING DATA

FINITE DIST. LATTICE    PARTIALLY ORDERED SET = POSET

CONVERSELY: $(**)$ $J(\mathcal{K}) \Rightarrow \mathcal{K}$ !   [S.M.R.]

POSET    F.D. LATTICE

$\forall L \in \mathcal{K},\ L = \dot{\bigcup}\{[K] \mid K \in J(\mathcal{K}),\ K \leq L\}$ (CHECK EXAMPLE

PARTITION $y_K = \begin{pmatrix} y_{\langle K\rangle} \\ \hdashline y_{[K]} \end{pmatrix}$ $\quad \left[ (y_1, y_2, y_3) = \begin{pmatrix} y_1 \\ y_2 \\ \hdashline y_3 \end{pmatrix} \right]$

$$\Sigma_K = \left(\begin{array}{c:c} \Sigma_{\langle K\rangle} & \Sigma_{\langle K]} \\ \hdashline \Sigma_{[K\rangle} & \Sigma_{[K]} \end{array}\right) \qquad \left[ \left(\begin{array}{cc:c} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23} \\ \hdashline \Sigma_{31} & \Sigma_{32} & \Sigma_{33} \end{array}\right) \right]$$

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THEOREM 2: If $\Sigma \in P^+_{\mathcal{K}}$ then

*① $|\Sigma| = \prod_{K \in J(\mathcal{K})} \left| \Sigma_{[K]\cdot\langle K\rangle} \right| \quad \le \prod_{K \in J(\mathcal{K})} |\Sigma_{[K]}|$

"HADAMARD for LATTICES"

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*② $y^t \Sigma^{-1} y = \sum_{K \in J(\mathcal{K})} \left( y_{[K]} - \underbrace{\beta_{[K]\cdot\langle K\rangle}}_{\Sigma_{[K\rangle}\Sigma^{-1}_{\langle K\rangle}} y_{\langle K\rangle} \right)^t \Sigma^{-1}_{[K]\cdot\langle K\rangle} (\cdots)$

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OBTAIN BASIC FACTORIZATIONS OF

LF & PARAMETER SPACE:

$\boxed{\Pi \text{ \& } \Sigma \text{ INDEXED BY } J(\mathcal{K})}$

$$\rightarrow (3)\quad f(x) = \prod_{K \in \mathcal{J}(\mathcal{K})} f\left(x_{[K]} \mid x_{\langle K \rangle}\right)$$

RECURSIVE FACTORIZAT.

LCI ⟺ "TDA

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$$\rightarrow (4)\quad \mathbb{R}^p \times \mathcal{P}^+_{\mathcal{K}} \xleftrightarrow{1-1}$$

$$(\mu, \Sigma)$$

$$\mathop{\times}_{K \in \mathcal{J}(\mathcal{K})} \left( \mathbb{R}^{[K]} \times M\left([K] \times \langle K \rangle\right) \times \mathcal{P}^+_{[K]} \right)$$

$$(\mu, \Sigma) \longleftrightarrow \underbrace{\left( \mu_{[K]\cdot\langle K \rangle},\ \beta_{[K]\cdot\langle K \rangle},\ \Sigma_{[K]\cdot\langle K \rangle} \right)}$$

$\equiv$ THE "$\mathcal{K}$-parameters" of $(\mu, \Sigma)$

---

EASILY OBTAIN MLE'S OF the $\mathcal{K}$-PARAM's, then OBTAIN $(\hat{\mu}, \hat{\Sigma})$ VIA a "Reconstruction algorithm" [later]

$(3) + (4) \Rightarrow$ LF FACTORIZATION.

**THEOREM 3.** UNDER THE RESTRICTION $\Sigma \in \mathcal{P}_{\mathcal{K}}$, $(\hat{\mu}, \hat{\Sigma})$ exists $\overset{\text{w.p.1}}{\Longleftrightarrow}$

$$(*) \quad \boxed{m_K^+ \geq |K| + 1} \quad \forall \text{ MAXIMAL ELTS } K \in J(\mathcal{K})$$

$\forall$ is unique in this (A)

---

HERE $m_K^+ = \Sigma(m_L \mid L \supseteq K)$

$+$ $m_L$ = # times $x_L$ IS OBSERVED

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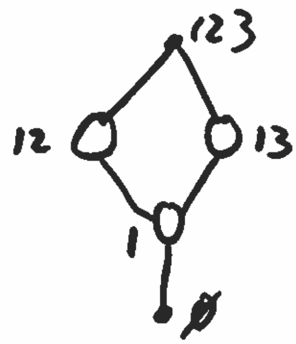
EXAMPLE ⓘ:

$\mathcal{K}$:



$\rightarrow$ [ELT'S IN $J(\mathcal{K})$ are CIRCLED "O"]

$J(\mathcal{K})$:

| | | | |
|---|---|---|---|
| $K =$ | 1 | 12 | 13 |
| $\langle K \rangle =$ | $\emptyset$ | 1 | 1 |
| $[K] =$ | 1 | 2 | 3 |

---

③ : $f = f(1 \mid \emptyset)\, f(2 \mid 1)\, f(3 \mid 1)$

④ : $\mathcal{K}$-param's: $(\mu_1, \Sigma_{11}), (\mu_{2\cdot 1}, \beta_{2\cdot 1}, \Sigma_{22\cdot 1}), (\mu_{3\cdot 1}, \beta_{3\cdot 1}, \Sigma_{33\cdot 1})$

---

IN GENERAL: $f(I) = \prod_{K \in J(\mathcal{K})} f([K] \mid \langle K \rangle)$ $(*)$

MAXIMAL ELTS OF $J(\mathcal{K})$ ARE: 12, 13

$\therefore$ MLE $(\hat{\mu}, \hat{\Sigma})$ exists $\iff$

$$(*) \quad \left\{ \begin{array}{l} m_{12}^{+} \equiv m_{12} + m_{123} \geq p_1 + p_2 + 1 \\ m_{13}^{+} \equiv m_{13} + m_{123} \geq p_1 + p_3 + 1 \end{array} \right\} \qquad (p_i = \dim(x_i))$$

EXAMPLE: MONOTONE CASE:

$\mathcal{K}$:

| $J(\mathcal{K})$: $K =$ | 1 | 12 | 123 |
|---|---|---|---|
| $\langle K \rangle =$ | $\emptyset$ | 1 | 12 |
| $[K] =$ | 1 | 2 | 3 |

(3) $f = f(1)\, f(2|1)\, f(3|12)$

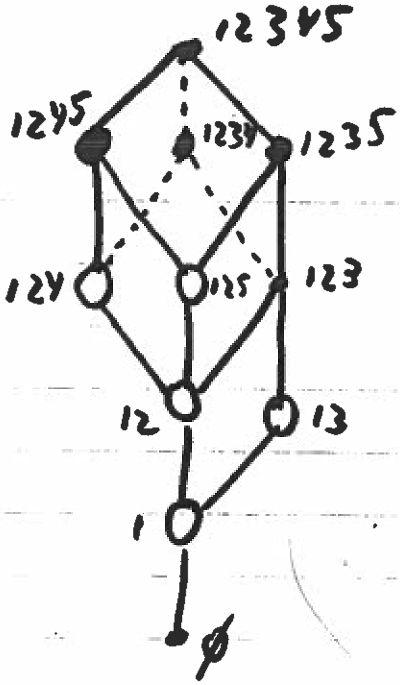
(4) $\mathcal{K}$-PARAMS: $(\mu_1, \Sigma_{11}), (\mu_{2\cdot1}, \beta_{2\cdot1}, \Sigma_{22\cdot1}), (\mu_{3\cdot12}, \beta_{3\cdot12}, \Sigma_{33\cdot12})$

MAXIMAL ELT OF $J(\mathcal{K})$ = 123

MLE EXISTS $\iff$ (*) $m_{123} \geq \underbrace{p_1 + p_2 + p_3}_{\equiv \dim(x)} + 1$

$\iff \boxed{m \geq p+1}$

EX. II :




$\Longrightarrow$

$$\boxed{\begin{array}{c|c} 3 \perp\!\!\!\perp 4 \perp\!\!\!\perp 5 & 12 \\ 2 \perp\!\!\!\perp 3 & 1 \end{array}}$$

$J(\mathcal{K})$:

| | | | | | |
|---|---|---|---|---|---|
| $K =$ | 1 | 12 | 13 | 124 | 125 |
| $\langle K \rangle =$ | $\emptyset$ | 1 | 1 | 12 | 12 |
| $[K] =$ | 1 | 2 | 3 | 4 | 5 |

---

(3) $f = f(1) f(2|1) f(3|1) f(4|12) f(5|12)$

(4) $\mathcal{H}$-PARAMS: $(\mu_1, \Sigma_{11}), (\mu_{2\cdot 1}, \beta_{2\cdot 1}, \Sigma_{22\cdot 1}), (\mu_{3\cdot 1}, \beta_{3\cdot 1}, \Sigma_{33\cdot 1})$

$(\mu_{4\cdot 12}, \beta_{4\cdot 12}, \Sigma_{44\cdot 12}), (\mu_{5\cdot 12}, \beta_{5\cdot 12}, \Sigma_{55\cdot 12})$

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MAXIMAL ELT'S OF $J(\mathcal{K})$: 13, 124, 125

$$(*) \begin{cases} m_{13}^{+} \equiv m_{13} + m_{123} + m_{1234} + m_{1235} + m_{12345} \geq p_1 + p_3 + 1 \\ m_{124}^{+} \equiv m_{124} + m_{1245} + m_{1234} + m_{12345} \geq p_1 + p_2 + p_4 + 1 \\ m_{125}^{+} \equiv m_{125} + m_{1245} + m_{1235} + m_{12345} \geq p_1 + p_2 + p_5 + 1 \end{cases}$$

## EXAMPLE: RUBIN (1974 JASA):

$$\mathcal{S} = \begin{array}{lcl} 1 & \to & f(1) \\ \ \ 2 & \to & f(2) \\ 1\ 2\ 3 & \to & f(1)\, f(2|1)\, f(3|12) \\ 1\ 2\ 3\ 4 & \to & f(1)\, f(2|1)\, f(3|12)\, f(4|123) \\ 1\ 2\ \ \ 4\ 5 & \to & f(1)\, \underline{\underline{f(2|1)}}\, \underline{\underline{f(4|12)}}\, f(5|124) \end{array}$$

CANNOT COMBINE

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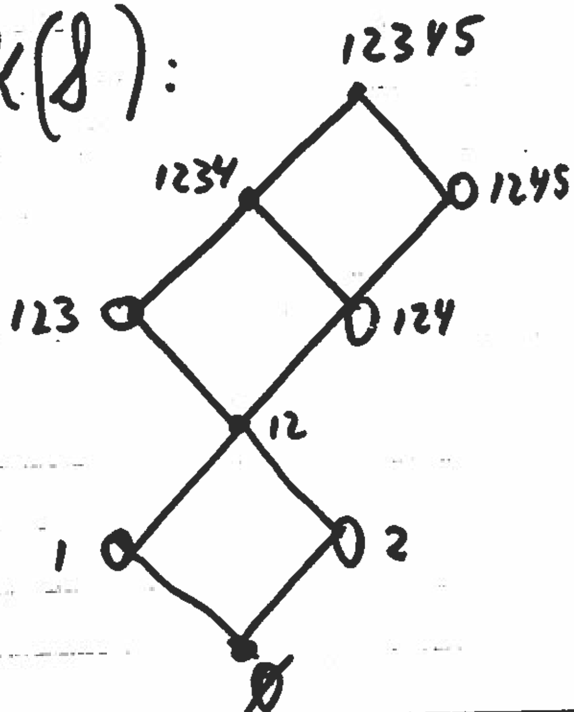
$\mathcal{K}(\mathcal{S})$:




$\Longrightarrow$

$$\boxed{\begin{array}{l} 3 \perp\!\!\!\perp 45 \mid 12 \\ 1 \perp\!\!\!\perp 2 \end{array}}$$

---

$\mathcal{J}(\mathcal{K})$:

| | | | | | |
|---|---|---|---|---|---|
| $K =$ | 1 | 2 | 123 | 124 | 1245 |
| $\langle K \rangle =$ | $\emptyset$ | $\emptyset$ | 12 | 12 | 124 |
| $[K] =$ | 1 | 2 | 3 | 4 | 5 |

5.

③ $f = f(1)\, f(2)\, f(3|12)\, f(4|12)\, f(5|124)$

④ $\mathcal{K}$-PARAM'S: $(\mu_1, \Sigma_{11})$, $(\mu_2, \Sigma_{22})$,

$(\mu_{3\cdot 12}, \beta_{3\cdot 12}, \Sigma_{33\cdot 12})$

$(\mu_{4\cdot 12}, \beta_{4\cdot 12}, \Sigma_{44\cdot 12})$

$(\mu_{5\cdot 124}, \beta_{5\cdot 124}, \Sigma_{55\cdot 124})$

---

MAXIMAL ELT'S OF $J(\mathcal{K})$: 123, 1245

$\therefore$ MLE $(\hat{\mu}, \hat{\Sigma})$ exists $\Leftrightarrow$

$$(*) \begin{cases} m^+_{123} \equiv m_{123} + m_{1234} + m_{12345} \geq p_1 + p_2 + p_3 + 1 \\ m^+_{1245} \equiv m_{1245} + m_{12345} \geq p_1 + p_2 + p_4 + p_5 + 1 \end{cases}$$

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NOTE: "FORGET ABOUT $\mathcal{S}$" UNTIL checking (*): $\boxed{m_K > 0 \Leftrightarrow K \in \mathcal{S}}$